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STABILITY OF COLUMNS WITH A SINGLE CRACK SUBJECTED TO FOLLOWER AND VERTICAL LOADS

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Abstract—A general flexibility matrix is developed which expresses the local flexibility of a beam of rectangular cross section with a single edge crack. The dominant term in this matrix is used to study the stability of the cracked column to follower and vertical loads. The kinetic method is used and eigencurves are developed to study the system stability.

For follower type loads, the Beck column, stability charts are given for several crack locations and sizes. Flutter type of instability is always encountered. For vertical loads the same type of analysis reveals the divergent type of instability, reported already in the literature. It was found that cracks can make this system have flutter type of instability for vertical loads.

1. INTRODUCTION

A crack on a structural member introduce a local flexibility which is a function of the crack depth. This flexibility changes the dynamic behaviour of the system and its stability characteristics. It must be emphasized that nonpropagating cracks are assumed in the sequel.

The local flexibility of a cracked beam was studied by Irwin[1] who related this flexibility (compliance) to the stress intensity factor.

The effect of the local flexibility of a cracked column upon its buckling load was studied by Liebowitz *et al.*[2, 3] and Okamura[4]. These authors identified the compliance of a cracked column to a bending moment. Rice and Levy[5] recognized the coupling between bending and extensional compliance of a cracked column in compression.

The effect of cracks upon the dynamic behaviour of cracked beams was studied by Dimarogonas[6] and Chondros and Dimarogonas[7, 8].

The effect of peripheral cracks upon the torsional vibration of a rod of circular cross-section was studied by Dimarogonas and Massouros[9].

In general, the above mentioned problems were considered as boundary value problems of elastic stability. Energy methods have not been used because the presence of a crack introduces discontinuities in the deformed shape of the column and it is difficult to find simple functions which will give sufficiently good approximation to the kinetic energy and the energy of elastic deformation. Therefore, for problems involving cracked members one has to resort to eigenvalue solutions of linear D.E. or numerical solutions of the nonlinear D.E.

For nonconservative systems the dynamic or kinetic approach is used[10–13]. Small vibrations about the zero equilibrium state are considered. The existence and the trend of such vibration can be directly interpreted as stable or unstable behavior of the system. Since this approach can be used also for conservative systems and facilitates the compliance description of local flexibilities due to cracks, it will be applied to the problems at hand.

To this end a cantilever column will be considered with vertical load and follower-type load (Beck column[14]), Fig. 1. The column is elastic, without imperfections other than a single edge crack transverse to the rectangular cross-section of the column. The crack has a length α on the section of height W.

2. THE LOCAL FLEXIBILITY MATRIX DUE TO CRACK

To study the effect of a crack upon the stability of an elastic structure, one has to establish the local stiffness or flexibility matrix of the cracked member under general loading. To this end a prismatic bar is considered with a crack of depth a along the y axis with a uniform depth along the z axis. The beam has height W and width B. The beam is loaded with axial force P_1, shear forces P_2, P_3 and bending moments P_4 and P_5, Fig. 2. Under general loading, the additional displacement u_i along the direction of force P_i due to the presence of the crack will be computed using Castigliano's theorem and by generalization of the Paris equation[10].

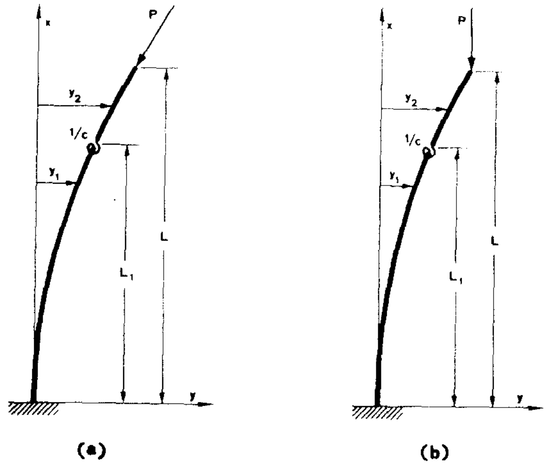


Fig. 1. (a) Cracked colum at $x = L_1$ with follower type load. (b) Cracked column at $x = L_1$ with vertical load. At both cases the crack is modelled as rotational spring of constant $1/c$.

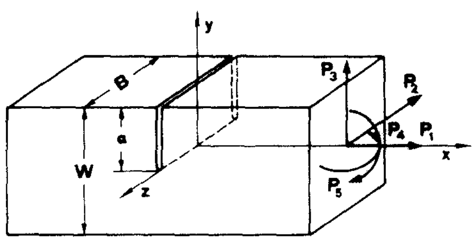


Fig. 2. Geometry of a rectangular cross-section beam with a single edge crack.

To this and, if U_T is the strain energy due to crack, Castigliano's theorem demands that the additional displacement is $u_i = \partial U_T/\partial P_i$ along the force P_i. The strain energy will have the form [1]. form [1].

$$U_T = \int_0^\alpha \frac{\partial U_T}{\partial \alpha} \, d\alpha = \int_0^\alpha J \, d\alpha \tag{1}$$

where $J = \partial U_T/\partial \alpha$ the strain energy density function. Therefore

$$u_i = \frac{\partial}{\partial P_i} \left[\int_0^\alpha J(\alpha) \, d\alpha \right] \text{(Paris Equation)}. \tag{2}$$

The flexibility influence coefficient c_{ij} will be, by definition.

$$c_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^\alpha J(\alpha) \, d\alpha. \tag{3}$$

The strain energy density function J has the general form

$$J = \frac{1}{E'} \left[\left(\sum_{l=1}^{5} K_{Il} \right)^2 + \left(\sum_{l=1}^{5} K_{IIl} \right)^2 + \kappa \left(\sum_{l=1}^{5} K_{IIIl} \right)^2 \right] \tag{4}$$

where, $E' = E$ for plane stress, $E' = E/(1 - \nu^2)$ for plane strain, $\kappa = 1 + \nu$, E and ν the Young modulus and the Poisson ratio respectively. Then, integrating along the cut (axis y),

$$c_{ij} = \frac{1}{E' b^2} \int_0^\alpha \left[\frac{\partial^2}{\partial P_i \partial P_j} \sum_m \int_0^B \left(e_m \sum_n K_{mn} \right)^2 dy \right] d\alpha \tag{5}$$

where $e_m = \kappa$ for $m = III$ and $e_m = 1$ for $m = I, II$. Furthermore, K_{mn} is the stress intensity factor of mode $m (m = I, II, III)$ due to the load $P_n (n = 1, 2, \ldots, 5)$ since the strain energy is additive. Adequate information is available for all k_{mn} except K_{I4} which will be assumed variable along the y axis and the plane solution will be used with the beam bending stress due to P_4 at location y. Then,

$$K_{I1} = \frac{P_1}{BW}\sqrt{\pi\alpha}\,F_1(\alpha/W) \tag{6}$$

$$K_{I5} = \frac{6P_5}{BW^2}\sqrt{\pi\alpha}\,F_2(\alpha/W) \tag{7}$$

$$K_{I4} = \frac{12P_4}{B^3W}\,y\sqrt{\pi\alpha}\,F_1(\alpha/W) \tag{8}$$

$$K_{I2} = K_{I3} = 0$$

$$K_{II1} = K_{II2} = K_{II4} = K_{II5} = 0$$

$$K_{II3} = \frac{2P_3}{BW\sqrt{\pi\alpha}}F_{II}(\alpha/W) \tag{9}$$

$$K_{III1} = K_{III3} = K_{III4} = K_{III5} = 0$$

$$K_{III2} = \frac{2P_2}{BW\sqrt{\pi\alpha}}F_{III}(\alpha/W) \tag{10}$$

where (10)

$$F_1(\alpha/W) = \sqrt{\frac{2W}{\pi\alpha}\tan\frac{\pi\alpha}{2W}}\,\frac{0.752 + 2.02(\alpha/W) + 0.37\left(1 - \sin\frac{\pi\alpha}{2W}\right)^3}{\cos\frac{\pi\alpha}{2W}}$$

$$F_2(\alpha/W) = \sqrt{\frac{2W}{\pi\alpha}\tan\frac{\pi\alpha}{2W}}\,\frac{0.923 + 0.199\left(1 - \sin\frac{\pi\alpha}{2W}\right)^4}{\cos\frac{\pi\alpha}{2W}}$$

$$F_{II}(\alpha/W) = \frac{1.30 - 0.65(\alpha/W) + 0.37(\alpha/W)^2 + 0.28(\alpha/W)^3}{\sqrt{1-\alpha/W}}$$

$$F_{III}(\alpha/W) = \sqrt{\frac{\pi\alpha}{W}\Big/\sin\frac{\pi\alpha}{W}}.$$

Using expressions (6) in eqn (5), we obtain

$$c_{11} = \frac{\Phi_{11}}{E'B^3W^2},\; c_{14} = \frac{12\Phi_{11}}{E'B^3W^2},\; c_{15} = \frac{6\Phi_{12}}{E'B^3W^3} \tag{11}$$

$$c_{44} = \frac{48\Phi_{11}}{E'B^4W^2},\; c_{45} = \frac{72\Phi_{12}}{E'B^3W^3},\; c_{55} = \frac{6\Phi_{22}}{E'B^3W^3} \tag{12}$$

$$c_{22} = \int_0^{\alpha}\frac{4\kappa F_{III}^2(\alpha/W)}{E'B^2W^2\pi(\alpha/W)}\,\mathrm{D}(\alpha/W) \tag{13}$$

$$c_{33} = \int_0^{\alpha} \frac{4F_{II}^2(\alpha/W)}{E'B^2W^2\pi(\alpha/W)} \mathrm{d}(\alpha/W) \tag{14}$$

$$\Phi_{ij} = \int_0^{\alpha} \frac{\pi\alpha}{W} F_i(\alpha/W)F_j(\alpha/W)\,\mathrm{d}\alpha.$$

The local flexibility matrix for the crack will have the form

$$C = \begin{bmatrix} c_{11} & 0 & 0 & c_{14} & c_{15} \\ 0 & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 & 0 \\ c_{41} & 0 & 0 & c_{44} & c_{45} \\ c_{51} & 0 & 0 & c_{54} & c_{55} \end{bmatrix}. \tag{15}$$

Due to reciprocity, the matrix C is symmetric. Some of the terms of matrix C can be found also in the literature. Okamura *et al.*[2–4] have computed c_{55}, Rice and Levy the terms c_{11}, c_{15}, c_{55}[5] and Dimarogonas and Massouros[9] the term c_{22}.

Except for stability analysis of cracked columns, the consept of the local flexibility analysis of cracked columns, the consept of the local flexibility due to cracks was used for other structures. Dimarogonas and Paipetis[11] studied the stability of a cracked shaft, while others[12, 13] reported on the stability of cracked shells due to external pressure.

The above developed local flexibility matrix due to cracks is based on an open crack. A single edge cracked column which is considered here, may exhibit different behavior at reversed bending due to closing of the crack. For this case, Gustafson[14] gives for reversed bending two stress intensity factors K^+ and K^- corresponding respectively to open and closing crack.

$$K^+ = F(\alpha/W)\sqrt{\pi\alpha} \quad K^- = G(\alpha/W)\sigma\sqrt{\pi\alpha}\sqrt{\sec\pi\left(\frac{2\alpha}{W}-1\right)} \tag{16}$$

where

$$F(\alpha/W) = 1.13 - 1.374(\alpha/W) + 5.749(\alpha/W)^2 - 4.464(\alpha/W)^3$$
$$+15.25(\alpha/W)^6 - 9.315(\alpha/W)^7$$
$$G(\alpha/W) = 0.544(2\alpha/W)(1 - W/2\alpha)^{3/2}.$$

Obviously, K^- exists only for $0.5 < \alpha/W < 1.0$.

Based on the above expressions, the terms c_{55} of the flexibility matrix, eqn (15), are

$$c_{55}^+ = \frac{W}{EI} C^+(\alpha/W),\ c_{55}^- = \frac{W}{EI} C^-(\alpha/W) \tag{17a, b}$$

where

$$C^+(\alpha/W) = 6\pi \int_0^{\alpha/W} \frac{\alpha}{W} F^2(\alpha/W)\,\mathrm{d}(\alpha/W) \tag{18a}$$

$$C^-(\alpha/W) = 0.8878\pi \int_0^{\alpha/W} \left(\frac{2\alpha}{W}-1\right)\sec\pi\left(\frac{2\alpha}{W}-1\right)\mathrm{d}(\alpha/W) \tag{18b}$$

obviously, $c_{55}^- = 0$ for $\alpha/W = 0.5$. The plot of nondimensional local stifness W/EIc is shown in Fig. 3, for the two modes.

For slender columns, only the torsional flexibility c_{55} is important to the problem[5]. Although the complete form of matrix C can be taken into account in what it follows, to simplify the presentation of results the term $c = c_{55}$ will be retained.

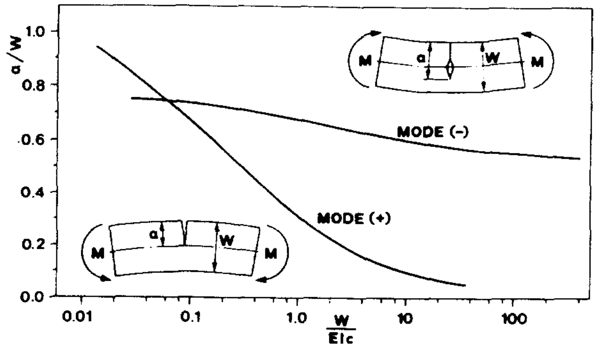


Fig. 3. Non dimensional compliance factor W/Eic versus non dimensional crack length a/W, for both bending modes.

ANALYSIS

The well known equation for small vibrations about the zero equilibrium state is

$$EIy_{,xxxx} + Py_{,xx} + \mu y_{,tt} = 0 \tag{19}$$

where EI is the flexural rigidity, P the axial load and μ the mass per unit length. The solution is assumed in the form

$$y(x, t) = Y(x)\,e^{i\Omega t}. \tag{20}$$

For real values of the frequency Ω the motion is simple harmonic. In general, Ω will have the form

$$\Omega = a + bi \tag{21}$$

and the resulting vibration will have increasing or decreasing amplitude depending on the sign of b.

Introducing the dimensionless quantities

$$\zeta = x/L,\ \lambda^2 = PL^2/EI,\ \omega_0 = \Omega L^2\sqrt{\mu/EI} \tag{22}$$

eqn (19)–(21) yield

$$Y^{IV}(\zeta) + \lambda^2 Y''(\zeta) - \omega_0^2 Y(\zeta) = 0 \tag{23}$$

with general solution

$$Y(\zeta) = A\sin\alpha_1\zeta + B\cos\alpha_1\zeta + C\sinh\alpha_2\zeta + D\cosh\alpha_2\zeta \tag{24}$$

where

$$\alpha_1 = [\sqrt{\omega_0^2 + \lambda^2/4} + \lambda^2/2]^{1/2}$$

$$\alpha_2 = [\sqrt{\omega_0^2 + \lambda^2/4} - \lambda^2/2]^{1/2}. \tag{25}$$

The crack divides the column into two parts with the same section properties. Therefore, eqn (24) will have the same α_1 and α_2 but different arbitrary constants which will be determined by the boundary conditions. Therefore,

$$Y_1(\zeta) = A_1\sin\alpha_1\zeta + B_1\cos\alpha_1\zeta + C_1\sinh\alpha_2\zeta + D_1\cosh\alpha_2\zeta$$

$$Y_2(\zeta) = A_2\sin\alpha_1\zeta + B_2\cos\alpha_1\zeta + C_2\sinh\alpha_2\zeta + D_2\cosh\alpha_2\zeta. \tag{26}$$

For the follower type load and for vertical load, the B.C. will be different:

(a) *Follower load*

$$Y_1(0) = 0 \quad Y_1'(0) = 0$$

$$Y_2''(1) = 0 \quad Y_2'''(1) = 0$$

$$KY_2'''(\beta) = Y_1'(\beta) - Y_2'(\beta)$$

$$Y_1(\beta) = Y_2(\beta),\ Y_1''(\beta) = Y_1''(\beta),\ Y_1'''(\beta) = Y_2'''(\beta) \tag{27}$$

where $\beta = L_1/L$ and $K = EIc/L$. The characteristic equations will be

$$\det A = 0 \tag{28}$$

Where the characteristic determinant A has the form:

$$\det A = \begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_1^2 \sin a_1 & -a_1^2 \cos a_1 & a_2^2 \sinh a_2 & a_2^2 \cosh a_2 \\ 0 & 0 & 0 & 0 & -a_1^3 \cos a_1 & a_1^3 \sin a_1 & a_2^3 \cosh a_2 & a_2^3 \sinh a_2 \\ \sin a_1\beta & \cos a_1\beta & \sinh a_2\beta & \cosh a_2\beta & -\sin a_1\beta & \cos a_1\beta & -\sinh a_2\beta & -\cosh a_2\beta \\ -a_1^2 \sin a_1\beta & -a_1^2 \cos a_1\beta & a_2^2 \sinh a_2\beta & a_2^2 \cosh a_2\beta & a_1^2 \sin a_1\beta & a_1^2 \cos a_1\beta & a_2^2 \sinh a_2\beta & -a_2^2 \cosh a_2\beta \\ -a_1^3 \cos a_1\beta & a_1^3 \sin a_1\beta & a_2^3 \cosh a_2\beta & a_2^3 \sinh a_2\beta & a_1^3 \cos a_1\beta & -a_1^3 \sin a_1\beta & -a_2^3 \cosh a_2\beta - & -a_2^3 \sinh a_2\beta \\ a_1 \cos a_1\beta & -a_1 \sin a_1\beta & a_2 \cosh a_2\beta & a_2 \cosh a_2\beta & \begin{matrix} a_1 \sin a_1\beta + \\ a_1^2\kappa \cos a_1\beta \end{matrix} & \begin{matrix} a_1 \sin a_1\beta + \\ a_1^2\kappa \cos a_1\beta \end{matrix} & \begin{matrix} -a_2 \cosh a_2\beta - \\ -a_2^2\kappa \sinh a_2\beta \end{matrix} & \begin{matrix} -a_2 \sinh a_2\beta - \\ a_2^2\kappa \cosh a_2\beta \end{matrix} \end{vmatrix}$$

(*b*) *Vertical load*

The B.C. are

$$Y_1(0) = 0,\ Y_1'(0) = 0$$

$$Y_2''(1) = 0,\ Y_2'''(1) = -\lambda^2 Y_2'(1) \tag{29}$$

$$KY_2''(\beta) = Y_1'(\beta) - Y_2'(\beta)$$

$$Y_1(\beta) = Y_2(\beta),\ Y_1''(\beta) = Y_2''(\beta),\ Y_1'''(\beta) = Y_2'''(\beta)$$

and the characteristic determinant A will have the form:

$$\det A = \begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_1^2 \sin a_1 & -a_1^2 \cos a_1 & a_2^2 \sin a_2 & a_2^2 \cosh a_2 \\ 0 & 0 & 0 & 0 & (a_1^3 + a_1\lambda^2) \cos a_1 & (a_1^3 - a_1\lambda^2) \sin a_1 & (a_2^3 + a_2\lambda^2) \cosh a_2 & (a_2^3 + a_2\lambda^2) \sinh a_2 \\ \sin a_1\beta & \cos a_1\beta & \sinh a_2\beta & \cosh a_2\beta & -\sin a_1\beta & -\cos a_1\beta & -\sinh a_1\beta & -\cosh a_2\beta \\ -a_1^2 \sin a_1\beta & -a_1^2 \cos a_1\beta & a_2^2 \sinh a_2\beta & a_2^2 \cosh a_2\beta & a_1^2 \sin a_1\beta & a_1^2 \cos a_1\beta & -a_2^2 \sinh a_2\beta & -a_2^2 \cosh a_2\beta \\ -a_1^3 \cos a_1\beta & -a_1^3 \sin a_1\beta & a_2^3 \cosh a_2\beta & a_2^3 \sinh a_2\beta & a_1^3 \cos a_1\beta & a^3 \sin a_1\beta & -a_2^3 \sinh a_2\beta & -a_2^3 \sinh a_2\beta \\ a_1 \cos a_1\beta & -a_1 \sin a_1\beta & a_2 \cosh a_2\beta & a_2 \sinh a_2\beta & \begin{matrix} -a_1 \cos a_1\beta + \\ +a_1^2\kappa \sin a_1\beta \end{matrix} & + \begin{matrix} a_1 \sin a_1\beta + \\ a_1^2\kappa \cos a_1\beta \end{matrix} & \begin{matrix} -a_2 \cosh a_2\beta - \\ -a_2^2\kappa \sinh a_2\beta \end{matrix} & \begin{matrix} -a_2^2\kappa \sinh a_2\beta - \\ -a_2^2\kappa \cosh a_2\beta \end{matrix} \end{vmatrix}$$

4. NUMERICAL RESULTS

The solution of eqn (28) for both types of loading consists of triagonalization of the determinant and finding the roots of the resulting polynomial. The solutions are found as functions $\omega_0 = f(\lambda^2)$ which are called eigencurves [12, 13, 15]. Parameters are the location of the crack, β, and the local flexibility of the crack K, related to the crack depth.

Two distinct types of stability behavior can be distinguished from the eigencurves (ω_0, λ^2). If the eigencurve intersects with the λ^2 axis, that means zero frequency at this load. This corresponds to the linear buckling load and will be called divergent type of bucking. If two sections of the eigencurve coalesce, this corresponds to the changing sign of b, eqn (21), which marks the threshold of instability and a vibrational behavior which will be called flutter type of buckling.

(a) *Follower load*

The eigencurves for follower type load are plotted in Fig. 4(a–d), corresponding respectively to $\beta = 0$, 1/4, 1/2, 3/4 for several crack flexibilities. Follower loads are nonconservative and they result in flutter type of instability in all cases. The closer the crack is to the clamped end the greater is the critical load. The same result follows the increasing crack depth. Instability is always of the flutter type

(b) *Vertical loads*

Eigencurves for vertical loads are given in Fig. 5 (a–d) for $\beta = 0, 1/4, 1/2, 3/4$ respectively, Vertical loads make the system conservative and lead to divergent type of instability, as predicted in the literature [2–5]. However, it is apparent in Fig. 4(c–d) that above certain value of the crack flecibility parameter, instability is of the flutter type. Eigencurves which begin at the lowest ω_0 correspond to the first eigenfrequency and to the lowest critical load. Higher eigencurves correspond to higher eigenmodes of instability.

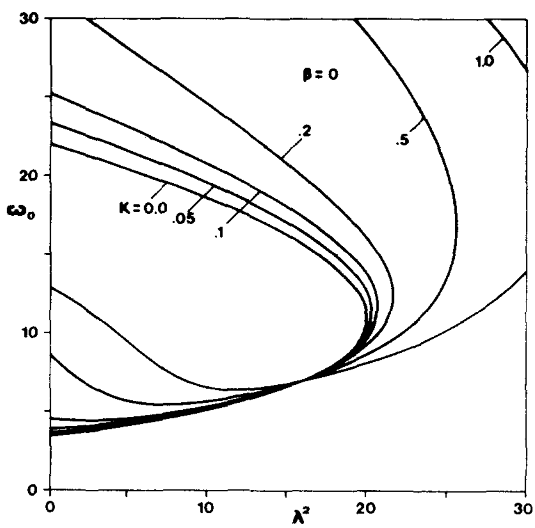


Fig. 4(a).

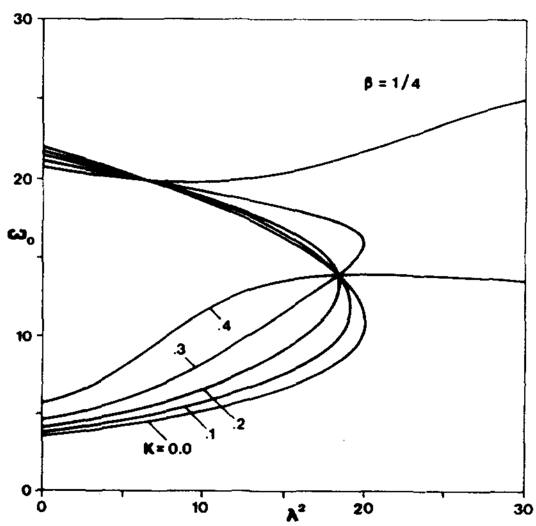


Fig. 4(b).

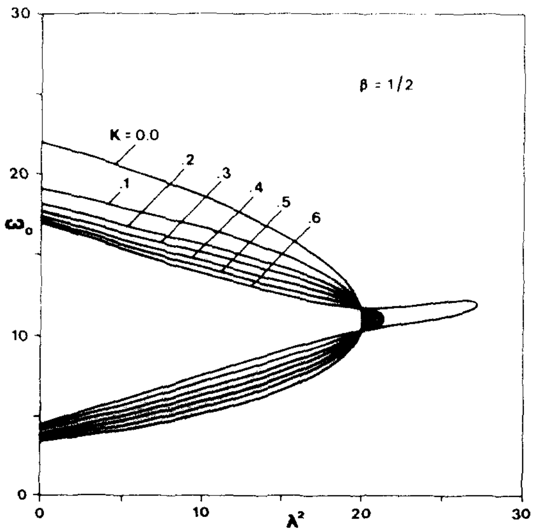


Fig. 4(c).

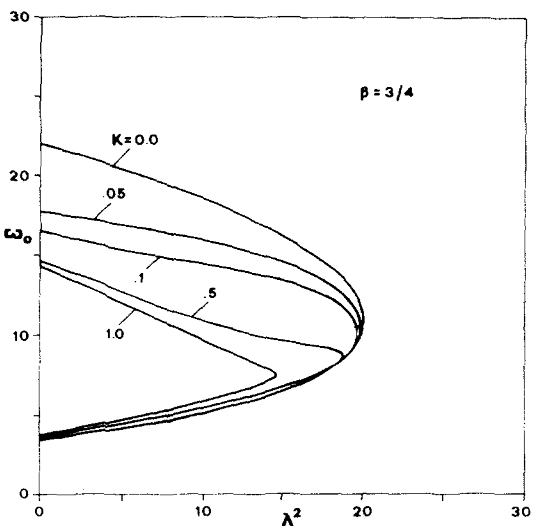
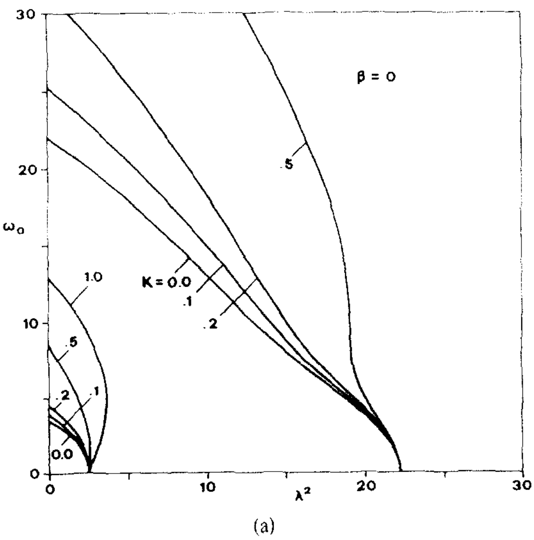
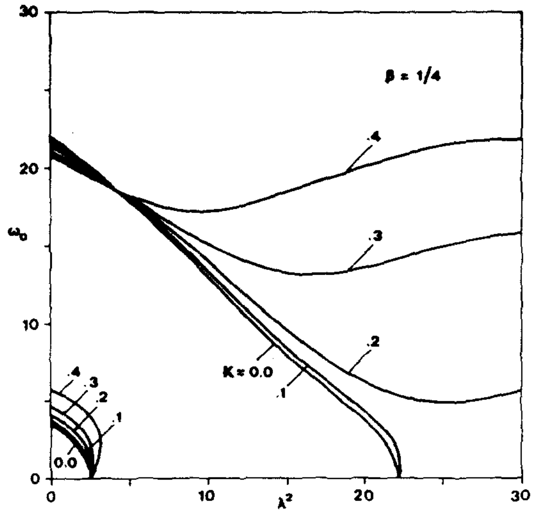


Fig. 4(d).

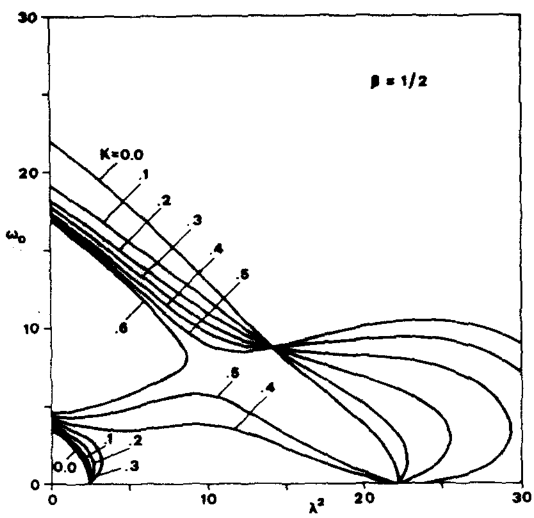
Fig. 4. (a) Eigencurves for follower type load, $\beta = 0$. (b) Eigencurves for follower type load, $\beta = 1/4$. (c) Eigencurves for follower type load, $\beta = 1/2$. (d) Eigencurves for follower type load, $\beta = 3/4$.



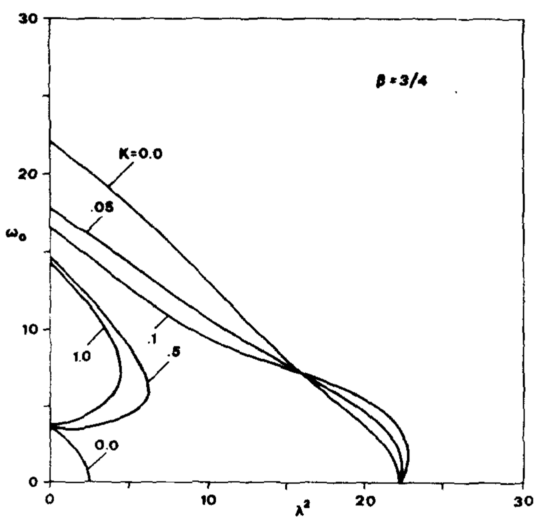
(a)



(b)



(c)



(d)

Fig. 5. (a) Eigencurves for vertical type load, $\beta = 0$. (b) Eigencurves for vertical type load, $\beta = 1/4$. (c) Eigencurves for vertical type load, $\beta = 1/2$. (d) Eigencurves for vertical type load, $\beta = 3/4$.

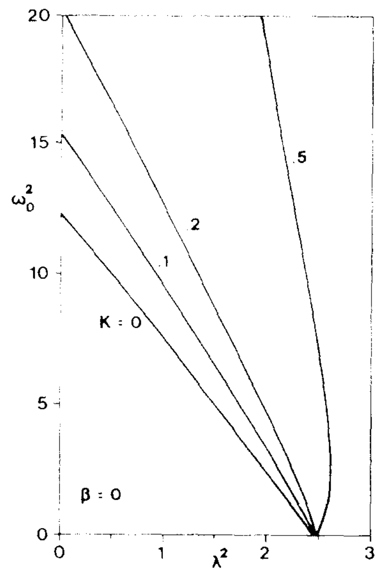


Fig. 6. Plot of eigencurves in (ω_0^2, λ^2) plane in the case of $\beta = 0$, which illustrates the deviation from linearity.

5. CONCLUSION

As expected, it was found that cracks can influence considerably the stability behavior of the Beck column or the vertically loaded column. The dynamic method was used and eigencurves were computed which can give divergent or flutter type of instability. The Beck column presents only the flutter type of instability. For high crack lengths the presence of the crack might in some cases increase the critical load.

For vertical loads, it was found that certain parameter constellations lead to flutter type of instability. Since the system is conservative, it is expected that for technical systems with damping the resulting vibration with be diminished with time.

Formally, this analysis is applicable only to notches or open cracks. Closing cracks exhibit different behavior for reversed bending at the vicinity of the crack. In addition for a closing crack, the column is loaded initially with closed crack and reacts as an uncracked column. When the load reaches the critical load for the uncracked column then the system moves, the crack opens and the column can sustain the load which corresponds to the critical load and behavior of the cracked column. Of course, random disturbances or initial imperfections might, make the system to move from a stable zero equilibrium state into an unstable open crack state. The disturbance needed is not infinitesimal. It has to be high enough to create tensile stresses in the vicinity of the crack. Therefore the uncracked equilibrium state might be stable in the small with a closed crack and jump saddently under a finite external disturbance into an unstable cracked state. This is particulary dangerous for engineering systems because it will be impulsive in character and can result in a sudden propagation of the crack.

An interesting observation can be made if the eigencurves are plotted in a (ω^2, λ^2) plane. As expected, the eigencurve for $K = 0$ is a straight line, Fig. 6. For cracked columns this relation is not a straight line. This renders inapplicable the method proposed in[19] for the experimental determination of the buckling load by way of vibration measurements. Moreover, it seems that the linear relation between load (λ^2) and natural frequency (ω_0^2) of the column which is well-known to exist for most boundary conditions, does not exist in the case of the cracked column.

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